GACS QUANTUM COMPLEXITY AND QUANTUM ENTROPY

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ABSTRACT

The development of quantum information theory motivated the extension to the quantum realm of notions from algorithmic complexity theory. Because of the structure of quantum mechanics, several inequivalent generalizations are possible. The same phenomenon characterizes the extension of the dynamical entropy of Kolmogorov and Sinai. In the following, we shall examine the relations between the quantum complexity introduced by Gacs and the quantum dynamical entropy proposed by Alicki and Fannes.

1. INTRODUCTION

Quantum information theory deals with what the transmission and manipulation of information when is the information carriers are quantum systems. Apart from its potential revolutionary applications to computational and criptographycal tasks [1], from a more abstract point of view, in view of teh framework offered by classical algorithmic compelxity theory [2], the use of quantum states raised the question of how complex they are and how complex is their dynamics [3,4,5,6,7]. From a statistical point of view, the von Neumann entropy rate of stationary quantum sources replaces the Shannon entropy rate of stationary classical sources. A stationary classical source can be viewed as a particular dynamical system where the dynamics is the shift along a classical spin chain and its entropy rate is the Kolmogorov-Sinai entropy of the shift. The Kolmogorov-Sinai entropy has several inequivalent extensions to quantum dynamical systems [8,9,10,11,12]; for quantum chains, namely one-dimensional lattices with a d-level quantum system at each site, one of these generalized dynamical entropies, the *Alicki-Fannes entropy* [9], differs from the von Neumann entropy rate by $\log d$. In the following we will relate this extra term to the *Gacs quantum complexity* [5] which is one among several proposals of how algorithmic complexity theory may be generalized to the quantum realm [3,4,5,6].

2. QUANTUM SOURCES

In quantum information theory, the central notion is that of a qubit, the most elementary quantum system described by a Hilbert space $\mathbb{H} = \mathbb{C}^2$ of dimension 2. The states of a qubit are of the form

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \qquad (1)$$

corresponding to classical bits 0 and 1 encoded by a spin $1/2$ pointing up and down along the z direction, or by orthogonal photon polarizations. However, the superposition principle tells us that physical states are also linear combinations

$$|\psi\rangle = a\,|0\rangle \,+\, b\,|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \ , \quad |a|^2 + |b|^2 = 1 \ , \quad (2)$$

and density matrices ρ. These are normalized ($\mathrm{Tr}\rho = 1$), positive 2×2 matrices:

$$\rho = \begin{pmatrix} r & \sigma \\ \sigma^* & 1-r \end{pmatrix} \ , \quad 0 \leq r \leq 1 \ , \ r(1-r) \leq |\sigma|^2 \ , \ (3)$$

such that, when spectralized, $\rho = r_1|r_1\rangle\langle r_1| + r_2|r_2\rangle\langle r_2|$, their eigenvalues $r_{1,2}$ are probabilities.

To a qubit there corresponds the algebra $\mathbf{M} = M_2$ of complex 2×2 matrices among which there are the qubit observables $A^\dagger = A$ (hermitian matrices) whose mean values with respect to ρ are given by

$$\langle A\rangle_\rho = \mathrm{Tr}(\rho\, A) = \sum_{i=1}^{2} r_i \langle r_i|A|r_i\rangle \ . \qquad (4)$$

The degree of mixedness of qubit states is measured by the von Neumann entropy

$$S(\rho) = -\mathrm{Tr}\rho \log \rho = -\sum_{i=1}^{2} r_i \log r_i \ , \qquad (5)$$

which corresponds to the Shannon entropy of the probability distribution given by the eigenvalues of ρ.

2.1. Quantum chains as quantum sources

In the following, we shall consider systems consisting of infinitely many qubits; for n of them the Hilbert space is $\mathbb{H} = \mathbb{C}^{2^n}$, the tensor product of the Hilbert spaces of the single qubits, their algebra is the tensor products of the single qubit matrix algebras,

$$M_{2^n} = M_2 \otimes M_2 \otimes \cdots M_2 =: \mathbf{M}_{[0,n-1]} \ ; \qquad (6)$$

while their state is a density matrix $\rho^{(n)}$ that is a positive matrix in $\mathbf{M}_{[0,n-1]}$ of trace 1.

The notation $\mathbf{M}_{[0,n-1]}$ indicates that the quantum system at hands can be thought as consisting of n qubits located at the integer sites of an initial segment of length n of a one-dimensional lattice. If the family of density matrices $\rho^{(n)}$ satisfies suitable compatibility conditions, letting $n \to \infty$ yields a *quantum chain* $\mathcal{M} = \lim_n \mathbf{M}_{[0,n-1]}$ equipped with a translation-invariant state $\omega = \lim_n \rho^{(n)}$ characterized by an *entropy rate*

$$s(\omega) = \lim_{n\to\infty} \frac{1}{n} S(\rho^{(n)}) . \qquad (7)$$

This quantum spin chain is a model of a stationary quantum information source: if, instead of the matrix algebras M_2, we considered diagonal algebras and equip the resulting chain with a compatible sequence of diagonal density matrices $\rho^{(n)}$, then we would get a classical stationary binary source.

2.2. Alicki-Fannes quantum dynamical entropy

For a classical source the (Shannon) entropy rate corresponds to the Kolmogorov-Sinai dynamical entropy of the shift along the strings emitted by the source; we shall now focus upon the Alicki-Fannes entropy which is one of its possible quantum generalizations. In the case of a quantum source, the dynamics, that is the shift along the chain, is represented by an automorphism Θ of the algebra $\mathcal{M}$ that moves single site algebras M_d one step to the right:

$$\Theta[\mathbf{M}_{[0,n-1]}] = \mathbf{M}_{[1,n]} . \qquad (8)$$

In quantum mechanics, measurement processes alter the system state on which they are performed and their effects are generically described by *partitions of unit*:

$$\mathcal{X} = \{X_i\}_{i=1}^p , \quad \sum_{i=1}^p X_i^\dagger X_i = 1 , \quad X_i \in \mathcal{M} . \qquad (9)$$

These operators provide a tomography of a density matrix $\rho \in M_2$ by means of an auxiliary density matrix $\rho[\mathcal{X}] \in M_p$ whose entries are:

$$\rho[\mathcal{X}]_{ij} = \mathrm{Tr}\Big(\rho\, X_j^\dagger\, X_i\Big) . \qquad (10)$$

For quantum spin chains, a useful partition is provided by the so-called *matrix units*, $u_{ij} \in M_2 = \mathbf{M}_0$:

$$\mathcal{U} = \left\{ u_{ij} = \frac{1}{\sqrt{d}} |i\rangle\langle j| \right\} , \quad u_{ij} u_{k\ell} = \delta_{jk} u_{i\ell} , \qquad (11)$$

where $\{|i\rangle\}_{i=1}^2$ in the orthonormal basis in (1). Under the dynamics, the matrix units evolve according to

$$\mathbf{M}_0 \ni u_{ij} \mapsto \Theta[u_{ij}] = 1 \otimes u_{ij} \in \mathbf{M}_{[0,1]} . \qquad (12)$$

After n dynamical steps, one constructs refined partitions of unit: $\mathcal{U}^{(n)} = \{u_{i^{(n)},j^{(n)}}\}$ whose 2^{2n} elements read

$$u_{i^{(n)},j^{(n)}} = u_{i_0 j_0} \otimes u_{i_1 j_1} \otimes \cdots u_{i_{n-1} j_{n-1}} . \qquad (13)$$

This set of 2^{2n} matrices provides a tomography of the state $\rho^{(n)}$, namely a $2^{2n} \times 2^{2n}$ *partition dependent* density matrix

$$\rho[\mathcal{U}^{(n)}] = \left[\mathrm{Tr}\Big(\rho^{(n)}\, u^\dagger_{i^{(n)} j^{(n)}} u_{k^{(n)} \ell^{(n)}}\Big) \right] , \qquad (14)$$

which describes how the dynamics influences the reconstruction of the system state in relation to a chosen partition of unit. Its matrix elements can be explicitly calculated using (11) and (13):

$$\mathrm{Tr}\left(\rho^{(n)}\, u_{j^{(n)} i^{(n)}} u_{k^{(n)} \ell^{(n)}}\right) =$$
$$= \delta_{i_0 k_0} \delta_{i_1 k_1} \cdot \delta_{i_{n-1} k_{n-1}} \langle \ell_0 \cdots \ell_{n-1} | \rho^{(n)} | j_0 \cdots j_{n-1} \rangle ,$$

so that the auxiliary density matrix reads

$$\rho[\mathcal{U}^{(n)}] = \frac{\mathbf{1}}{d^n} \otimes \rho^{(n)} . \qquad (15)$$

The latter has thus von Neumann entropy

$$S(\rho[\mathcal{U}^{(n)}]) = S(\rho^{(n)}) \,+\, n \,\log d , \qquad (16)$$

and entropy rate

$$h_\omega(\Theta, \mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} S(\rho[\mathcal{U}^{(n)}]) = s(\omega) + \log d . \quad (17)$$

This is the *Alicki-Fannes entropy* of the quantum chain as it is the maximum one can obtain by varying over all possible partitions of unit.

How can the extra term $\log d$ be interpreted? For this, in the following, we rely on quantum algorithmic complexity tools.

3. CLASSICAL ALGORITHMIC COMPLEXITY

While the entropy rate characterize the complexity of a classical binary source from a statistical point of view, the algorithmic complexity refers to individual strings $i^{(n)} \in \Omega_2^n = \{0,1\}^n$; it is the length $\ell(p)$ of the shortest binary program p which run by a prefix universal Turing machine (*UTM*) $\mathcal{T}$ outputs $i^{(n)}$:

$$K(i^{(n)}) = \min\left\{ \ell(p) \,:\, \mathcal{T}(p) = i^{(n)} \right\} . \qquad (18)$$

The algorithmic complexity of individual strings is intimately connected with the notion of *universal probability* of a string:

$$\mathbb{P}(i^{(n)}) := \sum_{p\,:\,\mathcal{T}[p]=i^{(n)}} 2^{-\ell(p)} . \qquad (19)$$

It turns out that, apart from an additive constant, independent of the string,

$$K(i^{(n)}) = -\log \mathbb{P}(i^{(n)}) \,+\, C . \qquad (20)$$

The universality of $\mathbb{P}$ relies upon the fact that, for any semi-computable semi-measure μ there exists a constant $C_\mu > 0$ dependent on μ only such that

$$\mu(i^{(n)}) \le C_\mu\, \mathbb{P}(i^{(n)}) . \qquad (21)$$

Individual and statistical complexity are related by the following two results: for any semi-computable probability $\pi^{(n)} = \{p(i^{(n)})\}$, the Shannon entropy is closed to the average Kolmogorov complexity:

$$H(\pi^{(n)}) = -\sum_{i^{(n)}\in\Omega_2^n} p(i^{(n)})\log p(i^{(n)}) \simeq \sum_{i^{(n)}\in\Omega_2^n} p(i^{(n)})\,K(i^{(n)})\,. \tag{22}$$

Furthermore, if the source is ergodic, in the limit of binary strings going to sequences the complexity per symbol equals the source entropy rate, for almost all of them:

$$k(i) := \lim_{n\to+\infty}\frac{1}{n}K(i^{(n)}) = h(\pi) = \lim_{n\to+\infty}\frac{1}{n}H(\pi^{(n)})\,. \tag{23}$$

4. GACS QUANTUM COMPLEXITY

Differently from other approaches that rely upon classical or quantum Turing machines, in [5] a quantum complexity is constructed by generalizing the notion of universal probability to that of *universal semi-density matrix*.

In order to do so, given a quantum system with Hilbert space $\mathbb{H}$, one relies on *elementary vectors* $|\Psi\rangle$ which are those with computable coefficients when expanded with respect to the basis obtained by tensor product of the vectors in (1). It is thus possible to describe them by means of a binary string of length n: $|\Psi\rangle \longleftrightarrow i_\Psi \in \Omega_2^n$ with associated universal probability $\mathbb{P}(i_\Psi)$. The universal semi-density matrix is then defined by a non-normalized convex combination of projectors onto all elementary vector states:

$$\mathbb{D} = \sum_\Psi \mathbb{P}(i_\Psi)\,|\Psi\rangle\langle\Psi|\,. \tag{24}$$

It is universal since, for any non-normalized density matrix $\rho \in \mathbf{M}$ whose entries are semi-computable, there exists a constant $C_\rho \geq 0$ depending only on ρ such that

$$\rho \leq C_\rho\,\mathbb{D}\,. \tag{25}$$

Then, one defines the *Gacs operator algorithmic complexity*, Gacs complexity for short, as

$$\kappa_q = -\log\mathbb{D}\,. \tag{26}$$

It turns out that the von Neumann entropy of any semi-computable semi-density matrix is close to the average Gacs complexity [5]:

$$S(\rho) \simeq \mathrm{Tr}(\rho\,\kappa_q)\,. \tag{27}$$

4.1. Gacs Complexity and Alicki-Fannes Entropy

In order to establish a relation between the Alicki-Fannes entropy and the Gacs complexity, it is necessary to examine closely the auxiliary matrix $\rho[\mathcal{U}^{(n)}]$ introduced in (15).

Let us first associate to the state on the quantum chain segment of length n, $M_{d^n} \ni \rho^{(n)} = \sum_i r_i^{(n)}\,|r_i^{(n)}\rangle\langle r_i^{(n)}|$ the vector state

$$|\sqrt{\rho^{(n)}}\rangle = \sum_i \sqrt{r_i^{(n)}}\,|r_i^{(n)}\rangle \otimes |r_i^{(n)}\rangle \tag{28}$$

in the doubled Hilbert space $\mathbb{C}^{2^n}\otimes\mathbb{C}^{2^n}$. Then, we associate to the auxiliary matrix $\rho[\mathcal{U}^{(n)}] \in M_{2^{2n}}$ the vector state

$$|\Psi[\mathcal{U}^{(n)}]\rangle = \sum_i \sum_{(k^{(n)}\ell^{(n)})} \sqrt{r_i^{(n)}}\,u_{k^{(n)}\ell^{(n)}}|r_i^{(n)}\rangle \otimes \otimes|r_i^{(n)}\rangle \otimes |(k^{(n)}\ell^{(n)})\rangle \tag{29}$$

in the Hilbert space $\mathbb{C}^{2^n}\otimes\mathbb{C}^{2^n}\otimes\mathbb{C}^{2^{2n}}$, where $|(k^{(n)}\ell^{(n)})\rangle$ enumerate the standard basis vectors in $\mathbb{C}^{2^{2n}}$.

Via this double purification, by partial tracing out, the first two Hilbert spaces first, and then the third one, one gets the marginal states

$$\mathrm{Tr}_{I,II}\Big(|\Psi[\mathcal{U}^{(n)}]\rangle\langle\Psi[\mathcal{U}^{(n)}]|\Big) = \rho[\mathcal{U}^{(n)}] \tag{30}$$

$$\mathrm{Tr}_{III}\Big(|\Psi[\mathcal{U}^{(n)}]\rangle\langle\Psi[\mathcal{U}^{(n)}]|\Big) = \sum_{i,j}\sqrt{r_i^{(n)}r_j^{(n)}}\,\times \times \sum_{(k^{(n)}\ell^{(n)})} u_{(k^{(n)}\ell^{(n)})}\,|r_i^{(n)}\rangle\langle r_j^{(n)}|\,u^\dagger_{(k^{(n)}\ell^{(n)})} \otimes \otimes|r_i^{(n)}\rangle\langle r_j^{(n)}| = R[\mathcal{U}^{(n)}]\,. \tag{31}$$

The previous two steps have first purified the density matrix for the initial length n segment of the quantum chain and then coupled it to an ancilla system described by a Hilbert space of dimension given by the number of elements in the partition of unit $\mathcal{U}^{(n)}$. As the marginal density matrices of a pure state have the same entropy, it turns out that

$$S(\rho[\mathcal{U}^{(n)}]) = S(R[\mathcal{U}^{(n)}]) = S(\rho^{(n)}) + n\,\log d \tag{32}$$

If we now consider the third system and take into account that the matrix units provide a computable partition and thus a computable density matrix $R[\mathcal{U}^{(n)}]$, relation (27) then states that the average Gacs complexity is close to the von Neumann entropy. Thus,

$$\mathrm{Tr}\Big(R[\mathcal{U}^{(n)}]\,\kappa_q^{(n)}\Big) \simeq S(R[\mathcal{U}^{(n)}]) = S(\rho^{(n)}) + n\,\log d\,. \tag{33}$$

5. CONCLUSION

While for classical dynamical systems there are essentially one dynamical entropy and one algorithmic complexity closely connected to each other, quantum mechanics offer different inequivalent options. We have considered the Alicki-Fannes entropy for a stationary quantum chain as an instance of quantum dynamical entropy; it is based on a tomographic reconstruction of the density matrix on longer and longer segments of the chain and gives an extra term with respect to the entropy rate of the chain. We have showed that this extra term can be related to the Gacs quantum complexity of the auxiliary system which provides the quantum tomography.

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