GENERALIZED NASH EQUILIBRIUM IMAGE DEBLURRING

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ABSTRACT

We compare two different formulations of the deblurring problem: one (variational) is defined by minimization of a single objective function and another one is based on a generalized Nash equilibrium balance of two objective functions. The latter results in the algorithm where the denoising and deblurring operations are decoupled. For image modeling we use the recent BM3D-frames. Simulation experiments show that the decoupled algorithm derived from the generalized Nash equilibrium formulation and using BM3D-frames demonstrates the best numerical and visual results and shows superiority with respect to the state of the art in the field.

1. INTRODUCTION

Image restoration from blurry and noisy observations is considered. Assuming a circular shift-invariant blur operator and additive zero-mean white Gaussian noise the observation model is expressed as

$$\mathbf{z} = \mathbf{A}\mathbf{y} + \sigma\varepsilon, \tag{1}$$

where $\mathbf{z}, \mathbf{y} \in \mathbb{R}^N$ are vectors representing the observed and true images, respectively, $\mathbf{A}$ is an $N \times N$ blur matrix, $\varepsilon \sim \mathcal{N}(\mathbf{0}_{N\times 1}, \mathbf{I}_{N\times N})$ is a vector of i.i.d. Gaussian random variables, and σ is the standard deviation of the noise. The deblurring problem is to reconstruct $\mathbf{y}$ from $\mathbf{z}$.

Image modeling lies at the core of image reconstruction problems. Recent trends are concentrated on *sparse representation* techniques, where the image is assumed to be defined as a combination of few *atomic* functions taken from a certain *dictionary*. It follows that the image can be parameterized and approximated locally or nonlocally by these functions. To enable sparse approximations, the dictionary should be rich enough to grasp all variety of images. Clearly, the classical orthonormal bases are too limited for this task, and one needs to consider overcomplete systems with a number of elements essentially larger than the dimensionality of approximated images. *Frames* are generalization of the concept of basis to the case when atomic functions are linearly dependent and form overcomplete systems. There is a vast amount of literature devoted to the sparsity based models and methods. An excellent introduction and overview of this area can be found in the recent book [1].

The block-matching 3D (BM3D) image denoising, originated in [2], is formalized in [3] and [4] in terms of the overcomplete sparse frame representation. The analysis and synthesis developed in BM3D are interpreted as a general sparse image modeling applicable to various image processing problems.

In this paper we discuss two different variational formulations of the image deblurring proposed in our recent papers [3] and [4]: single objective function optimization vs. fixed point of two objective functions (generalized Nash equilibrium). The latter approach results in the algorithm where denoising and deblurring operations are decoupled. It is shown by simulation experiments that the best image reconstruction both visually and numerically is obtained by the algorithm based on this decoupling. To the best of our knowledge, this algorithm provides results which are the state-of-art in the field.

1.1. BM3D-frame image modeling

It has been shown in [3] that provided a fixed grouping the BM3D analysis/synthesis can be given in the matrix form linking the image $\mathbf{y} \in \mathbb{R}^N$ and its groupwise spectrum vector $\boldsymbol{\omega} \in \mathbb{R}^M$, $M \gg N$, by the forward and backward transforms

$$\boldsymbol{\omega} = \mathbf{\Phi} \cdot \mathbf{y}, \mathbf{y} = \mathbf{\Psi} \cdot \boldsymbol{\omega}. \tag{2}$$

It is proved in [3] that the matrices $\mathbf{\Phi}^T\mathbf{\Phi}$ and $\mathbf{\Psi}\mathbf{\Psi}^T$ are diagonal with positive items; $\mathbf{\Psi}\mathbf{\Phi} = \mathbf{I}_{N\times N}$. The last formula enables perfect reconstruction of the image $\mathbf{y}$ from its groupwise spectrum $\boldsymbol{\omega}$. It is shown also that $\mathbf{\Phi}$ and $\mathbf{\Psi}^T$ are full column rank matrices. The rows of $\mathbf{\Phi}$ constitute a frame in $\mathbb{R}^N$, and the columns of the $\mathbf{\Psi}$ constitute a frame in $\mathbb{R}^N$ dual to $\mathbf{\Phi}$. These frames are non-tight, $\mathbf{\Phi}^T \cdot \mathbf{\Phi} \neq \lambda \mathbf{I}_{N\times N}$ and $\mathbf{\Psi}^T \cdot \mathbf{\Psi} \neq \lambda \mathbf{I}_{N\times N}$, $\lambda > 0$. If the group weights in BM3D synthesis [2] are equal to 1, then $\mathbf{\Psi} = (\mathbf{\Phi^T\Phi})^{-1}\mathbf{\Phi}^T$, however in general, $\mathbf{\Psi} \neq (\mathbf{\Phi^T\Phi})^{-1}\mathbf{\Phi}^T$.

Once BM3D groups are defined, the operators $\mathbf{\Phi}$, $\mathbf{\Phi}^T$, $\mathbf{\Psi}$ and $\mathbf{\Psi}^T$ can be implemented efficiently since all of them perform groupwise separable 3-D transforms. To build the groups the block matching (grouping) procedure from [2] is used. The *BM3D*-frames are nonlocal and data adaptive, which make them quite different from the other popular frames used for image modeling.

1.2. Variational formulations for deblurring

The *analysis* (using the analysis matrix $\mathbf{\Phi}$) and *synthesis* (using the synthesis matrix $\mathbf{\Psi}$) variational image reconstructions are conventional for an overcomplete image modeling [1]. For a Gaussian noise these reconstructions can be given in the form of constrained optimization, respectively, for *analysis*

$$\hat{\mathbf{y}} = \arg\min_{\mathbf{y}} \{\frac{1}{2\mu} \|\mathbf{z} - \mathbf{A}\mathbf{y}\|_2^2 + \tau \cdot \|\boldsymbol{\omega}\|_p \mid \boldsymbol{\omega} = \mathbf{\Phi}\mathbf{y}\}, \tag{3}$$

and *synthesis*

$$\hat{\boldsymbol{\omega}} = \arg\min_{\boldsymbol{\omega}}\{\frac{1}{2\mu}\left\|\mathbf{z}-\mathbf{A}\mathbf{y}\right\|_2^2 + \tau\cdot\left\|\boldsymbol{\omega}\right\|_p \mid \mathbf{y}=\boldsymbol{\Psi}\boldsymbol{\omega}\}. \tag{4}$$

These two formulations are studied thoroughly in literature assuming that the frames are tight, $\boldsymbol{\Phi}^T\cdot\boldsymbol{\Phi}=\mathbf{I}_{N\times N}$, $\boldsymbol{\Psi}=\boldsymbol{\Phi}^T$. The last formula says that the analysis matrix $\boldsymbol{\Phi}$ defines completely the synthesis one and vice versa. For the non-tight BM3D-frames these matrices do not define each other, and for image reconstruction we need to use both the analysis and synthesis operators.

In this way we arrive to the *combined analysis/synthesis* formulation of image reconstruction [4]:

$$\begin{aligned}(\hat{\boldsymbol{\omega}},\hat{\mathbf{y}}) &= \arg\min_{\boldsymbol{\omega},\mathbf{y}}\{\frac{1}{2\mu}\left\|\mathbf{z}-\mathbf{A}\mathbf{y}\right\|_2^2+\tau\cdot\left\|\boldsymbol{\omega}\right\|_p \mid \\ \boldsymbol{\omega} &= \boldsymbol{\Phi}\mathbf{y},\ \mathbf{y}=\boldsymbol{\Psi}\boldsymbol{\omega}\},\end{aligned} \tag{5}$$

where the analysis and synthesis links between the image and spectrum are considered as constraints.

For $p=1$ and $p=0$ (3)-(5) are defining l_2-l_1 and l_2-l_0 optimization problems, respectively.

Let us replace the constraints in (3)-(5) by the quadratic penalties with positive weights γ_s. In this way for (5) we arrive to the objective function

$$\begin{aligned}\mathcal{L}(\mathbf{y},\boldsymbol{\omega}) &= \frac{1}{2\mu}\left\|\mathbf{z}-\mathbf{A}\mathbf{y}\right\|_2^2+\tau\cdot\left\|\boldsymbol{\omega}\right\|_p + \\ & \frac{1}{2\gamma_1}\left\|\boldsymbol{\omega}-\boldsymbol{\Phi}\mathbf{y}\right\|_2^2+\frac{1}{2\gamma_2}\left\|\mathbf{y}-\boldsymbol{\Psi}\boldsymbol{\omega}\right\|_2^2.\end{aligned} \tag{6}$$

This $\mathcal{L}(\mathbf{y},\boldsymbol{\omega})$ is universal in the sense, that with $\gamma_1\to\infty$ and $\gamma_2\to 0$ it corresponds to the *synthesis* approach

$$\hat{\boldsymbol{\omega}} = \arg\min_{\boldsymbol{\omega}}\{\frac{1}{2\mu}\left\|\mathbf{z}-\mathbf{A}\boldsymbol{\Psi}\boldsymbol{\omega}\right\|_2^2+\tau\cdot\left\|\boldsymbol{\omega}\right\|_p\}, \tag{7}$$

and with $\gamma_2\to\infty$ and $\gamma_1\to 0$ it corresponds to the *analysis* approach

$$\hat{\mathbf{y}} = \arg\min_{\mathbf{y}}\{\frac{1}{2\mu}\left\|\mathbf{z}-\mathbf{A}\mathbf{y}\right\|_2^2+\tau\cdot\left\|\boldsymbol{\Phi}\mathbf{y}\right\|_p\}. \tag{8}$$

With finite γ_1,γ_2 it defines a *combined synthesis/analysis* approach.

1.3. Generalized Nash equilibrium (GNEP) problems

Let us briefly recall the basic formulations of the GNEP [5]. Formally, the GNEP consists of N players, each player v controlling the variables $x^v\in\mathbb{R}^{n_v}$. We denote by $\mathbf{x}$ the vector formed by all these variables: $\mathbf{x}=\begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^N\end{pmatrix}$, which has dimension $n=\sum_{v=1}^{N}$, and by $\mathbf{x}^{-v}$ the vector formed by all the players' decision variables except those of player v. To emphasize the $v-th$ player's variables within $\mathbf{x}$, we sometimes write $(x^v\ ,\mathbf{x}^{-v})$ instead of $\mathbf{x}$.

Each player has an objective function $f_v:\mathbb{R}^{n_v}\to\mathbb{R}$ that depends on both his own variables x^v as well as on the variables $\mathbf{x}^{-v}$ of all other players. This mapping f_v is often called the utility function of player v, sometimes also the payoff function or loss function, depending on the particular application in which the GNEP arises.

Furthermore, each player's strategy must belong to a set $X_v\subseteq\mathbb{R}^{n_v}$ that depends on the rival players' strategies and that we call the feasible set or strategy space of player v. The aim of player v, given the other players' strategies $\mathbf{x}^{-v}$, is to choose a strategy x^v that solves the minimization problem

$$\min_{x^v} f_v(x^v,\mathbf{x}^{-v}) \text{ subject to } x^v\in X_v(\mathbf{x}^{-v}). \tag{9}$$

For any x^v, the solution set of problem (9) is denoted by $S_v(\mathbf{x}^{-v})$. The GNEP is the problem of finding a vector $\bar{\mathbf{x}}$ such that

$$\bar{x}^v\in S_v(\mathbf{x}^{-v}) \text{ for all } v=1,...N.$$

Such a point $\bar{\mathbf{x}}$ is called *Generalized Nash Equilibrium* or, more simply, a solution of the GNEP. A point $\bar{\mathbf{x}}$ is therefore an equilibrium if no player can decrease his objective function by changing unilaterally x^v to any other feasible point. If we denote by $S(\mathbf{x})=\cap_{v=1}^{N}S_v(\mathbf{x}^{-v})$, we see that we can say that $\bar{\mathbf{x}}$ is a solution if $\bar{\mathbf{x}}\in S(\mathbf{x})$, i.e. if $\bar{\mathbf{x}}$ is a fixed point of the point-to-set mapping S. If the feasible sets $X_v(\mathbf{x}^{-v})$ do not depend on the rival players' strategies, so we have $X_v(\mathbf{x}^{-v})=X_v$ for some set $X_v\subseteq\mathbb{R}^{n_v}$ and all $v=1,...,N$, the GNEP reduces to the standard Nash equilibrium problem (NEP for short).

For non-empty and convex $X_v(\mathbf{x}^{-v})$ the existence of the solution can be guaranteed provided some reasonable assumptions (e.g. Theorem 6 in [5]).

The sets $X_v(\mathbf{x}^{-v})$ can be given by equality or inequality constrains, for instance, as

$$X_v(\mathbf{x}^{-v})=\{x^v:\ g^v(x^v,\mathbf{x}^{-v})\leq 0\}. \tag{10}$$

Recall, that a Pareto vector optimization is different from the Nash equilibrium by a simultaneous optimization on $\mathbf{x}$. A vector $\tilde{\mathbf{x}}$, $\tilde{x}^v\in X_v(\tilde{\mathbf{x}}^{-v})$, is Pareto optimal, if there exists no other vector $\mathbf{y}$ such that

$$f_v(\mathbf{y})\leq f_v(\tilde{\mathbf{x}}), \text{ for all } v=1,...,N$$

and $f_i(\mathbf{y})<f_i(\tilde{\mathbf{x}})$ for at least one i, provided that $y^v\in X_v(\mathbf{y}^{-v})$. It is known that the GNEP and Pareto optimization, in general, give different solutions.

1.4. Deblurring as generalized Nash equilibrium problem

Let us formulate the deblurring for observations (1) as the following GNEP problem [3]:

$$\begin{cases}\hat{\mathbf{y}}=\arg\min_{\mathbf{y}}\frac{1}{2\sigma^2}\left\|\mathbf{z}-\mathbf{A}\mathbf{y}\right\|_2^2 \text{subjectto} \left\|\mathbf{y}-\boldsymbol{\Psi}\,\hat{\boldsymbol{\omega}}\right\|_2^2\leq\varepsilon_1, \\ \hat{\boldsymbol{\omega}}=\arg\min_{\boldsymbol{\omega}}\tau\cdot\left\|\boldsymbol{\omega}\right\|_p \text{ subject to } \left\|\boldsymbol{\omega}-\boldsymbol{\Phi}\hat{\mathbf{y}}\right\|_2^2\leq\varepsilon_2,\end{cases} \tag{11}$$

where $\varepsilon_1,\varepsilon_2>0$, and the inequality constrains $\left\|\mathbf{y}-\boldsymbol{\Psi}\,\hat{\boldsymbol{\omega}}\right\|_2^2\leq\varepsilon_1$ and $\left\|\boldsymbol{\omega}-\boldsymbol{\Phi}\hat{\mathbf{y}}\right\|_2^2\leq\varepsilon_2$ relax the equalities (2).

Two groups of variables $\mathbf{y}$ and $\boldsymbol{\omega}$ define two players in the formulation (9) with the corresponding objective and restriction functions. For the algorithm development we replace (11) by unconstrained optimization problems:

$$\begin{cases}\hat{\mathbf{y}}=\arg\min_{\mathbf{y}}\mathcal{L}_1(\mathbf{y},\hat{\boldsymbol{\omega}}) \\ \hat{\boldsymbol{\omega}}=\arg\min_{\boldsymbol{\omega}}\mathcal{L}_2(\hat{\mathbf{y}},\boldsymbol{\omega})\end{cases}, \tag{12}$$

Scenario	PSF	σ^2
1	$1/\left(1+\|\|x\|\|_2^2\right), \|x_i\| \leq 7,$	2
2	$1/\left(1+\|\|x\|\|_2^2\right), \|x_i\| \leq 7,$	8
3	9×9 uniform	≈ 0.3
4	$[1\,4\,6\,4\,1]^T\,[1\,4\,6\,4\,1]/256$	49
5	Gaussian with $std = 1.6$	4
6	Gaussian with $std = 0.4$	64

Table 1. Blur PSF and noise variance used in each scenario.

where

$$\mathcal{L}_1(\mathbf{y}, \boldsymbol{\omega}) = \tag{13}$$
$$\frac{1}{2\sigma^2}\left\|\mathbf{z} - \mathbf{A}\mathbf{y}\right\|_2^2 + \frac{1}{2\gamma}\left\|\mathbf{y} - \boldsymbol{\Psi}\boldsymbol{\omega}\right\|_2^2,$$
$$\mathcal{L}_2(\mathbf{y}, \boldsymbol{\omega}) = \tau \cdot \left\|\boldsymbol{\omega}\right\|_p + \frac{1}{2\xi}\left\|\boldsymbol{\omega} - \boldsymbol{\Phi}\mathbf{y}\right\|_2^2. \tag{14}$$

A solution $(\hat{\mathbf{y}}, \hat{\boldsymbol{\omega}})$ of (12) is a fixed point or Nash equilibrium of the two objective functions $\mathcal{L}_1$ and $\mathcal{L}_2$. Minimization of the quadratic $\mathcal{L}_1$ on $\mathbf{y}$ results in a linear solution which is a regularized inverse of the blur operator $\mathbf{A}$. Minimization of the non-quadratic $\mathcal{L}_2$ on $\boldsymbol{\omega}$ results in a nonlinear hard- or soft-thresholding solution filtering noise.

1.5. IDD-BM3D algorithm

The proposed iterative algorithm follows from the alternating solution of (12):

$$\begin{cases} \mathbf{y}_{t+1} = \arg\min\limits_{\mathbf{y}} \mathcal{L}_1(\mathbf{y}, \boldsymbol{\omega}_t) \\ \boldsymbol{\omega}_{t+1} = \arg\min\limits_{\boldsymbol{\omega}} \mathcal{L}_2\left(\mathbf{y}_{t+1}, \boldsymbol{\omega}\right) \end{cases}, \; t = 0, 1, \ldots . \tag{15}$$

We call this algorithm Iterative Decoupled Deblurring BM3D (IDD-BM3D). Details of the fast implementation of the algorithm using FFT calculations as well as the converge statement can be seen in [3].

1.6. Simulation experiments

Here some test-results from [3] are presented. We consider six deblurring scenarios used as the benchmarks in many publications. The blur point spread function (PSF) $h\left(x_1, x_2\right)$ and the variance of the noise σ^2 for each scenario are summarized in Table 1.

Four proposed algorithms, namely: analysis-based, synthesis-based, combined and IDD-BM3D are evaluated in the scheme with the soft thresholding (the penalty norm is l_1) and unit group weights ($g_r = 1$). Additionally, the IDD-BM3D algorithm is tested with the adaptive group weights using the soft and hard thresholdings.

In Table 2 we present improvement of signal-to-noise ratio (ISNR) values achieved by each algorithm for the *Cameraman* image. From these values we can conclude that the synthesis-based algorithm performs essentially worse than the IDD-BM3D algorithm, with the analysis-based algorithm being in-between. Comparing the last two rows, we conclude that hard thresholding enables better results than the soft thresholding, and combined with the adaptive weights it provides the best results among the considered algorithms.

In the experiments with the combined analysis/synthesis algorithm we optimize the parameters of the objective function (6). Nevertheless, the results obtained by this algorithm are not better and only close to those obtained by the analysis algorithm. The comparison is definitely in favor of the IDD-BM3D algorithm based on GNEP. We wish to note also that optimization of the parameters in the objective function (6) for the combined algorithm actually gives the results close to those which can be obtained using the Pareto optimization. Thus, the simulation results demonstrate also the advantage of GNEP versus the Pareto optimization.

The experiments with the IDD-BM3D algorithm can be reproduced using the Matlab program available as a part of the BM3D package[1].

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Method			Scenario					
	Thresh.	Weights g_r	1	2	3	4	5	6
Cameraman (256x256)								
BSNR			31.87	25.85	40.00	18.53	29.19	17.76
Input PSNR			22.23	22.16	20.76	24.62	23.36	29.82
Synthesis	soft	unit	6.30	4.60	7.88	2.06	2.98	2.84
Analysis	soft	unit	7.88	5.75	9.22	3.00	3.67	3.92
IDD-BM3D	soft	unit	8.17	6.17	9.38	3.17	3.83	4.12
IDD-BM3D	soft	adaptive	8.41	6.41	9.59	3.38	3.98	4.14
IDD-BM3D	hard	adaptive	**8.85**	**7.12**	**10.45**	**3.98**	**4.31**	**4.89**

Table 2. Comparison of the output ISNR [dB] of the proposed deblurring algorithms. Row corresponding to "Input PSNR" contain PSNR [dB] of the input blurry images. Blurred signal-to-noise ratio (BSNR) is defined as $10log_{10}\left(var\left(\mathbf{Ay}\right)/N\sigma^2\right)$, where $var()$ is the variance.

Figure 1. Deblurring of the *Cameraman* image, scenario 3. From left to right and from top to bottom are presented zoomed fragments of the following images: original, blurred noisy, reconstructed by CGMK [6] (ISNR 9.15), L0-AbS [7] (ISNR 9.10), DEB-BM3D [8] (ISNR 8.34) and by proposed IDD-BM3D method (ISNR 10.45).